Mathematics MDM4U Data Management
– Independent Study, Conditional Probability

Let us begin today's lesson with a quick review. Previously, we talked about **independent events** and **dependent events**. An **event** is a subset of the **sample space**. Independent events are events which do not affect the probability of each other. For example, we say Event A and Event B are independent if and only the probability of occurrence of A does not influence the probability of occurrence of B. Algebraically, we define Event A and Event B being independent if and only if …

$$P(A \cap B) = P(A)P(B) \quad [1]$$

On the other hand, dependent events are event in which the probability of occurrence of one affects the probability of occurrence of the other. For example, **mutually exclusive events** are necessarily dependent. Dependent events are algebraically defined as …

$$P(A \cap B) \neq P(A)P(B) \quad [2]$$

Today we will analyze the probability of dependent events. Specifically, we are going to study the probability of an event given a particular condition. For example, it is easy to understand that the probability of drawing a heart from a standard deck is 50%. However, what about this question?

What is the probability of drawing a heart given that the card is red?

Using just intuition, we may immediately notice that the probability will be 25%. "Why?", you might ask? Since we are told that the card is red, we know that the card is either a heart or a diamond. In addition, we know that the probability of getting a heart is the same as the probability of getting a diamond. As a result, 50% of 50% is 25%. Although it is nice that we may solve simple problems involving **conditional probability** using just intuition and a bit of logic, we definitely need a more systematic approach for the harder problems. Algebraically, we are going to define conditional probability as the probability of Event A given Event B (will occur). Algebraically, we define the probability of A given B to be …

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad [3]$$

Let us redo the above question, but using formula [3]. The question "what is the probability of getting a heart given the selected card is red?" can be reworded into "what is the probability of A given B", where …

A = selecting a heart
B = selecting a red card

By applying formula [3], we have …

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(red\ and\ heart)}{P(red)} = \frac{13/52}{26/52} = \frac{13}{26} = 50\%$$

Let us consider some other, perhaps more difficult, examples. This question will involve organized counting techniques.

Sample Problem: Suppose a box contains 4 identical blue balls, 3 identical green balls, and 3 identical red balls. If two balls are randomly selected at once, what is the probability that …

a) the selected balls are different in color given at least one of them is blue?
b) the selected balls contain at least one red, given both are different in colors?

Full Solution a)

Suppose we let A = *the selected balls are different in color*, and let B = *the selected balls have at least one in blue*. Please note that B is the equivalent to say that *the selected balls are either blue-red, blue-green, or blue-blue*. The event $A \cap B$ can be interpreted as *the selected balls are different in color and has exactly one blue*. Alternatively, this is the same as saying *the selected balls are either blue-red, or blue-green*. Using formula [3], we have …

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(blue-red\ or\ blue-green)}{P(exactly\ one\ blue)}$$

$$= \frac{\dfrac{\binom{4}{1}\binom{3}{1}+\binom{4}{1}\binom{3}{1}}{\binom{10}{2}}}{\dfrac{\binom{4}{1}\binom{3}{1}+\binom{4}{1}\binom{3}{1}+\binom{4}{2}}{\binom{10}{2}}} = \frac{12+12}{12+12+6} = \frac{24}{30} = 80\%$$

Full Solution b)

Suppose we let A = *the selected balls have at least one red*, and let B = *the selected balls are different in colors*. We may reword B as *the selected balls are not red-red, blue-blue, nor green-green*. We may also interpret $A \cap B$ as *the selected balls are either red-green, or red-blue*. As a result, using formula [3] will give us …

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(red-green\ or\ red-blue)}{P(different\ in\ colors)}$$

$$= \frac{\dfrac{\binom{3}{1}\binom{3}{1}+\binom{3}{1}\binom{4}{1}}{\binom{10}{2}}}{\dfrac{\binom{10}{2}-\binom{4}{2}-\binom{3}{2}-\binom{3}{2}}{\binom{10}{2}}} = \frac{9+12}{45-6-3-3} = \frac{21}{33} = \frac{3}{11} \approx 27.3\%$$